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Higher-order level correlations in integrable quantum systems

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Abstract. In this paper we study the second, third and fourth cumulants of the distribution of the number of energy levels in an interval containing $\bar{n}$ levels on an average. The distribution function, which also depends on the distance N (measured in units of average level spacings) above the ground state, can be obtained either from the exact quantum levels or from the semiclassical approximation to the level density. Either of the two methods yields the same values for the cumulants. In two limits the cumulants have been obtained analytically. First, for small $\bar{n}$ the functional dependence of the cumulants is obtained using generalisations of Berry's semiclassical sum rule for the two-point function. We find that in this limit the first four cumulants are equal to the average number of levels $\bar{n}$ in the interval. Second, for larger values of $\bar{n}$ the functional dependence can be obtained by taking into account only the slowly oscillating contributions to the cumulants of the semiclassical level density. For the second cumulant we obtain an analytical expression when $\bar{n} \gg 1$. We find that the correction term to $\bar{n}$ is proportional to $-\bar{n}^2/\sqrt{N}$ (this correction does not show up in the Δ_3 statistic). This cumulant saturates at a value of $\sim\sqrt{N}$ at a scale of $\bar{n} \sim \sqrt{N}$. Analytical expressions for the third and fourth cumulants are obtained only for distances larger than $\bar{n} \gg N^{1/3}$ and $\bar{n} \gg N^{3/8}$, respectively. After an initial rise to far above the Poisson limit their values oscillate with a frequency and amplitude of $\sim\sqrt{N}$.

1. Introduction

In generic Hamiltonian systems the solutions of the equations of motion show an enormous richness in structure as we increase the interaction between the degrees of freedom (see, e.g., Lichtenberg and Lieberman 1983). This inspired many authors to explore quantum systems as a function of the interaction between the degrees of freedom. In recent years most work has been devoted to the study of the eigenvalues and eigenfunctions of time-independent Hamiltonian systems with two degrees of freedom and to the study of systems with one degree of freedom driven by a time-dependent external force. We will only discuss the investigation of eigenvalues and dispose of the other topics with the remark that many interesting results have been obtained (see, for example, Casati *et al* 1986, Chang and Shi 1985, Feingold *et al* 1985, Fishman *et al* 1982, Heller 1984, Israilev 1986, José and Cordery 1986, Kuś *et al* 1986, Shapiro and Goelman 1984, Blümel and Smilansky 1985, Seligman *et al* 1986).

Most studies of the eigenvalues involve their statistical analysis. The most important results have been discovered by numerical investigations of systems with two degrees of freedom. They can be summarised as follows. For integrable systems the eigenvalues

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are essentially distributed as independent random variables, whereas for classically chaotic systems the eigenvalues are distributed as the eigenvalues of the invariant random matrix ensembles. There is a smooth transition between the integrable and the completely chaotic regime (see Berry 1981, Berry and Robnik 1986, Bohigas *et al* 1984, Delande and Gay 1986, Haller *et al* 1984, Ishikawa and Yukawa 1985, Robnik and Berry 1986, Seligman *et al* 1985a, b, Seligman and Verbaarschot 1985a, b). Recently, similar observations have been made for quasi-energies (Israilev 1986, José and Cordery 1986, Kuś *et al* 1986). For the integrable case we want to mention the following generic deviations from the general result. First, the eigenvalues are distributed as independent random variables only for distances (measured in units of the average level spacing) short compared to the square root of the number of levels above the ground state. For distances large compared to the square root of the number of levels above the ground state the fluctuations of the eigenvalues saturate (see Seligman *et al* 1985b, Seligman and Verbaarschot 1986, Casati *et al* 1985 and Berry 1985). Second, when the potential has a strong harmonic term the distribution of the eigenvalues becomes very complicated. For a discussion of the 'harmonic oscillator anomaly' we refer to Berry and Tabor (1976) and Bohigas *et al* (1983).

Most spectra have been analysed by means of the nearest-neighbour spacing distribution and the Δ_3 statistic. The latter statistic is a measure for the average deviation of the number of levels in an interval containing a given average number of levels. The average can either refer to an average over the spectrum or to an average over the ensemble or to a combination of both. Recently, several authors have employed other statistics to analyse spectra (see Bohigas *et al* 1985 and Roman *et al* 1986). We mention the cumulants of the distribution of the number of levels in a interval containing a given number of levels on an average. The second cumulant (also called number variance) is closely related to the Δ_3 statistic but does yield some extra information. The third and fourth cumulant provide us with another independent test for the conjectures put forward by the aforementioned numerical work. It is generally believed that in the integrable case as well, the third and fourth moments tend to the Poisson limit for distances small compared to the square root of the number of levels above the ground state. However, it is not known in which way the saturation of the fluctuations manifests itself in these correlation functions. The investigation of this question is the main objective of this paper.

The theoretical understanding of numerical studies in terms of the semiclassical expansion for the level density is mainly due to the work of Berry (1985). For integrable systems he was able to explain the saturation of the fluctuations. For classically chaotic systems he succeeded in deriving the correct logarithmic dependence of the Δ_3 statistic. As in our previous paper on the Δ_3 statistic, we will base this work on the ideas put forward by Berry (1985). We will carry out our investigations on scaling integrable systems (i.e. integrable systems with a homogeneous polynomial as a potential). This has the advantage that many calculations can be carried out explicitly, both analytically and numerically.

The organisation of this paper is follows. In § 2 we review the results for the semiclassical level density of scale invariant systems, as obtained in earlier works. This fixes our notation and allows a concise formulation in the following sections. In this section we also explain that the number variance contains more information than the Δ_3 statistic. In § 3 we study the short-range behaviour of the cumulants of the distribution of the number of levels. We will find new sum rules for the expansion coefficients and the phases of the periodic orbit sum for the level density. In § 4 we

give analytical expressions for the long-range behaviour of the cumulants of the distribution of the number of levels. Numerical results are presented in § 5 and concluding remarks are made in § 6.

2. The semiclassical level density

In this section we give the semiclassical expansions for the level density of a two-dimensional integrable scaling system. For the derivations we refer to Seligman and Verbaarschot (1987). Two-dimensional integrable scaling systems are defined by the Hamiltonian (see Landau and Lifshitz 1969)

$$H = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + \gamma_1 x_1^{2q} + \gamma_2 x_2^{2q} \tag{2.1}$$

where the x_i are the coordinates and the p_i are the momenta. The γ_i are arbitrary constants. In the WKB approximation (which appears to be extremely accurate) the energy levels are given by (see Seligman and Verbaarschot 1987)

$$E_{mn} = \alpha(m+\tfrac{1}{2})^p + \beta(n+\tfrac{1}{2})^p \tag{2.2}$$

where p is related to the order of the homogeneous polynomial by

$$p = 2q/(q+1). \tag{2.3}$$

The use of the levels E_{mn} in (2.2) instead of the exact quantum mechanical levels does not affect any of our results (see Seligman and Verbaarschot (1987) for a discussion of this point). Since multiplication of α and β by the same constant leads to a trivial rescaling of the levels the product of α and β can be chosen to be equal to 1. Via a Poisson resummation and a stationary phase approximation (see Berry and Mount 1972, Berry and Tabor 1977a, b, Richens and Berry 1982, Seligman and Verbaarschot 1987) it is possible to derive the semiclassical periodic orbit sum for the level density from equation (2.2). In terms of the integrated level density the result is given by

$$N_T(E) = \bar{N}(E) + N_P(E) + N_{osc}(N(E)) \tag{2.4}$$

where

$$\bar{N}(E) = \frac{1}{2p} E^{2/p} B\left(\frac{1}{p}, \frac{1}{p}\right) \tag{2.5a}$$

$$N_P(E) = \alpha^{1/p} \sum_k [E - \alpha(k+\tfrac{1}{2})^p]^{1/p}$$

$$+ \alpha^{-1/p} \sum_l [E - \alpha^{-1}(l+\tfrac{1}{2})^p]^{1/p} - \frac{1}{2p} E^{2/p} B\left(\frac{1}{p}, \frac{1}{p}\right) \tag{2.5b}$$

$$N_{osc}(N) = \int_{-\infty}^{N} \mathrm{d}N\, N^{-1/4} \sum_{\boldsymbol{M}}{}' A_{\boldsymbol{M}} \exp[\mathrm{i}\, \mathrm{sgn}(M_1)(\sqrt{N} S_{\boldsymbol{M}} - \tfrac{1}{4}\pi)]. \tag{2.5c}$$

The sum in equation (2.5*b*) is over all k and l larger than or equal to zero for which the expression below the power $1/p$ is positive. The argument of N_{osc} is the 'average' number of levels above the ground state. The primed summation in (2.5*c*) is defined as the sum over all M_1 and M_2 different from zero and with equal sign. The amplitudes

A_M and the actions S_M are given by

$$A_M = \frac{1}{p\sqrt{p-1}}\left(\frac{2p}{B(1/p,1/p)}\right)^{1/4}(M_1M_2)^{(2-p)/2(p-1)} \times\left[\left(\frac{M_1}{\alpha^{1/p}}\right)^{p/(p-1)}+\left(\frac{M_2}{\alpha^{-1/p}}\right)^{p/(p-1)}\right]^{(p-3)/2p} \tag{2.6}$$

$$S_M = 2\pi\left(\frac{2p}{B(1/p,1/p)}\right)^{1/2}\left[\left(\frac{M_1}{\alpha^{1/p}}\right)^{p/(p-1)}+\left(\frac{M_2}{\alpha^{-1/p}}\right)^{p/(p-1)}\right]^{(p-1)/p}.$$

The first term in (2.4) is the Thomas-Fermi approximation to the level density (this term contains the contributions of the periodic orbits of length zero). For a billiard the second term gives rise to the perimeter term in the level density (this term contains the contributions of the periodic orbits of length zero in one and only one degree of freedom). The last term in (2.4) is the genuinely oscillating contribution (both M_1 and M_2 are different from zero).

In our numerical analysis we compare correlation functions obtained from the level sequence E_{mn} (see equation (2.2)) and correlation functions obtained from $N_{\rm osc}$. In order to make the comparison unambiguous we have to normalise the level sequence to unit level spacing by using the slowly varying terms in the level density. The normalised sequence E'_{mn} can be obtained from E_{mn} by the following prescription:

$$E'_{mn} = \int_{-\infty}^{E_{mn}} [\bar{\rho}(E)+\rho_P(E)]\,\mathrm{d}E \tag{2.7}$$

where $\bar{\rho}$ and ρ_P are the level densities corresponding to (2.5*a*) and (2.5*b*).

The statistics we will study are the first four cumulants $\Sigma_p(\bar{n}, N)$. They are defined as the expectation values of the cumulants of the distribution of the number of levels $\mathcal{N}_N(\bar{n})$ in an interval of length $\bar{n}$:

$$\Sigma_p(\bar{n}, N) = \langle \mathcal{N}_N^p(\bar{n})\rangle_c \tag{2.8}$$

where $\langle\ \rangle_c$ denotes a local average over an interval in the spectrum that is small compared to the total number of levels above the ground state and which is large compared to $\bar{n}$. The subscript c (from connected) means that all correlations that factorise into products of lower-order moments have been subtracted. When the eigenvalues are uncorrelated (the Poisson limit) the distribution of $\mathcal{N}_n(\bar{n})$ is given by $(\bar{n}^{\mathcal{N}_N(\bar{n})}/\mathcal{N}_N(\bar{n})!)\exp(-\bar{n})$ (for a level sequence with average level density equal to unity). In this case all cumulants are equal to $\bar{n}$.

The Δ_3 statistic is related to the number variance by (Pandey 1979, Brody *et al* 1982)

$$\Delta_3(\bar{n}) = \frac{2}{\bar{n}^4}\int_0^{\bar{n}} \mathrm{d}s(\bar{n}^3 - 2\bar{n}^2 s + s^3)\Sigma_2(s, N). \tag{2.9}$$

Under this transformation a monomial s^p is transformed as

$$s^p \to \bar{n}^p \frac{2-p}{(p+1)(p+2)(p+4)} \tag{2.10}$$

from which we conclude that the integral transform (2.9) has a non-trivial kernel. The term in $\Sigma_2(s, N)$ proportional to s^2 will not be present in the $\Delta_3(\bar{n})$ statistic! Moreover, all higher-order terms are suppressed by large numerical factors. Consequently the number variance contains more information than the Δ_3 statistic.

For a further discussion of the cumulants and their relation to level correlation functions we refer to Brody *et al* (1982) and Pandey (1979). Because the pth cumulant is a p-point measure we will also refer to it as a p-point function.

The distribution of $\mathcal{N}_N(\bar{n})$ can be obtained from the level sequence E'_{mn} by counting the number of levels in consecutive intervals of length $\bar{n}$. A semiclassical approximation for $\mathcal{N}_N(\bar{n})$ is obtained by the evaluation of the difference $N_{\rm osc}(N+\bar{n})-N_{\rm osc}(N)$. We will assume that $\bar{n}$ is much smaller than N (the number of levels the interval lies above the ground state). This is a necessary condition for spectral averaging. In this case the square root occurring in the exponent of the integrand of (2.5*c*) can be linearised. This renders the integration trivial. The result is given by

$$N_{sm}(\bar{n}, N) = 8N^{1/4} \sum_{M_1=1}^{\infty} \sum_{M_2=1}^{\infty} \frac{A_M}{S_M} \cos\left(\sqrt{N} S_M + \frac{\bar{n}}{4\sqrt{N}} S_M - \tfrac{1}{4}\pi\right) \sin\left(\frac{\bar{n}}{4\sqrt{N}} S_M\right) \tag{2.11}$$

where A_M and S_M are defined in (2.6). N is the number of levels above the ground state given by the sum $\bar{N}(E)+N_P(E)$. To relate $\bar{n}$ to an energy difference it suffices to use $\bar{\rho}$. The density ρ_P is $O(E^{-1/p-1})$ smaller than the leading term $\bar{\rho}$.

Without the oscillating term the summation over $\boldsymbol{M}$ in (2.11) diverges as $|\boldsymbol{M}|^{1/2}$. Therefore we can expect non-neglible contributions from large values of $|\boldsymbol{M}|$. Because important contributions stem from a region with $S_M \approx 2\pi\sqrt{N}/\bar{n}$ this plays a role for small values of $\bar{n}$ in particular. In practice the numerical evaluation of (2.9) will be inhibited for small values of $\bar{n}$.

Higher-order terms in the stationary phase approximation for $\rho_{\rm osc}$ do not contribute to $N_{\rm SM}(\bar{n}, N)$. In the first place, they are of higher order in $N^{-1/2}$. In the second place, they converge better with respect to the summation over $\boldsymbol{M}$.

In the next section we will study the second, third and fourth cumulants of $N(\bar{n})$.

3. Short-range behaviour of the number moments

Berry (1985) was able to derive the short-range behaviour of the Δ_3 statistic by using a semiclassical sum rule. Analogous sum rules are needed to obtain the short-range behaviour of the number moments. The sum rules express the fact that for very short distances the only correlations are self-correlations.

The sum rules can be derived with the help of the trace of the Green function G_η. In the diagonal representation G_η is given by

$$G_\eta(E) = \sum_n \frac{1}{E - E_n + \mathrm{i}\eta} \qquad \eta > 0 \tag{3.1}$$

where the E_k are the eigenvalues of the Hamiltonian and η is an infinitesimal quantity. For the eigenvalues given in (2.2) the sum in (3.1) is divergent. However, the fluctuating part of the Green function (i.e. $G_\eta(E)-\langle G_\eta(E)\rangle_c$) is finite. Below the subscript c indicates that we consider this combination only. We first consider the correlation of the Green functions for finite but very small values of η. For the two-point function we consider the product

$$\langle G_{\eta_1}(E) G_{-\eta_2}(E)\rangle_c = \sum_{k_1 k_2} \left\langle \frac{1}{E - E_{k_1} + \mathrm{i}\eta_1} \frac{1}{E - E_{k_2} - \mathrm{i}\eta_2} \right\rangle_c \tag{3.2}$$

where $\langle\ \rangle_c$ is defined below equation (2.8) [e.g. the local average may be defined as $\langle f\rangle = (1/\pi)\int \mathrm{d}E f(E)\Gamma/(E^2+\Gamma^2)$]. Products in which both factors have their poles on the same side of the real axis are uninteresting; the average over E can be factorised in averages over the single factors† and we will find no connected contributions. By

† $\langle G_\eta(E) G_\eta(E)\rangle = (1/\pi)\int \mathrm{d}E\, G_\eta(E) G_\eta(E)\Gamma(E+\mathrm{i}\Gamma)^{-1}(E-\mathrm{i}\Gamma)^{-1} = G(E-\mathrm{i}\Gamma)G(E-\mathrm{i}\Gamma)$
$= \langle G_\eta(E)\rangle\langle G_\eta(E)\rangle$.

using similar arguments we find that in equation (3.2) the terms with $k_1 \neq k_2$ do not contribute to the connected part of the average when $\eta_i \to 0$. The only non-trivial correlations arise from terms with $k_1 = k_2$ and with the poles located on opposite sides of the real axis. After making a partial fractions decomposition for the diagonal terms in the summation in (3.2) we obtain

$$\langle G_{\eta_1}(E) G_{-\eta_2}(E)\rangle_c = 2\pi\bar{\rho}/(\eta_1+\eta_2) \tag{3.3}$$

where the average level density $\bar{\rho}$ is given by

$$\bar{\rho} = -(1/\pi)\,\mathrm{Im}\langle G_\eta\rangle. \tag{3.4}$$

For the three-point function we have to evaluate the product of three Green functions. As in the case of the two-point function, we find non-trivial contributions for $\eta_i \to 0$ only when the poles are not on the same side of the real axis. We choose the location of the poles as follows ($\eta_i > 0$):

$$\langle G_{\eta_1}(E) G_{-\eta_2}(E) G_{-\eta_3}(E)\rangle_c = \sum_{k_1 k_2 k_3} \left\langle \frac{1}{E-E_{k_1}+i\eta_1}\,\frac{1}{E-E_{k_2}-i\eta_2}\,\frac{1}{E-E_{k_3}-i\eta_3}\right\rangle_c. \tag{3.5}$$

When η_i are small compared to the average level spacing only the terms with $k_1 = k_2 = k_3$ are important. After making two consecutive partial fractions decompositions we obtain

$$\langle G_{\eta_1}(E) G_{-\eta_2}(E) G_{-\eta_3}(E)\rangle_c = 2\pi i\bar{\rho}\,\frac{1}{(\eta_1+\eta_3)(\eta_2+\eta_3)}. \tag{3.6}$$

A similar identity can be derived for all higher-order correlation functions. We only quote the result for the four-point function. In this case we find three essentially different configurations for the location of the poles in the complex plane. In the first group all the the poles are on the same side of the real axis. Such terms do not yield connected contributions. The second group contains terms with three poles on one side and one pole on the other side of the real axis. The third group contains terms with two poles on either side of the real axis. Depending on the signs preceding the $i\eta_i$ we find the following result for the four-point function:

$$\langle G_{\eta_1}(E) G_{\eta_2}(E) G_{-\eta_3}(E) G_{-\eta_4}(E)\rangle_c = 2\pi\bar{\rho}\,\frac{\eta_1+\eta_2+\eta_3+\eta_4}{(\eta_1+\eta_3)(\eta_2+\eta_3)(\eta_1+\eta_4)(\eta_2+\eta_4)} \tag{3.7a}$$

$$\langle G_{\eta_1}(E) G_{-\eta_2}(E) G_{-\eta_3}(E) G_{-\eta_4}(E)\rangle_c = -2\pi\bar{\rho}\,\frac{1}{(\eta_1+\eta_2)(\eta_1+\eta_3)(\eta_1+\eta_4)}. \tag{3.7b}$$

We want to stress that the three- and four-point functions depend only on the average level density and the η_i.

Berry (1985) has given higher-order sum rules in terms of the level density. Since they involve particular combinations of the Green functions (see (3.4)), they constitute a special case of the sum rules presented in this work.

On the other hand, we can calculate the three- and four-point functions from the semiclassical approximations for the Green functions. The semiclassical expansion for G_η can be derived in the standard Feynman path formalism for non-relativistic quantum mechanics. The result which is an analytic continuation of ρ_{osc} is given by (see Berry and Mount 1972, Gutzwiller 1967, 1970, 1971)

$$\begin{aligned} G_\eta &= -2\pi i\bar{\rho} N^{-1/4} \sum_M{}^{+} A_M \exp(i\sqrt{N} S_M - \tfrac{1}{4}\pi i)\exp\left(-\frac{\eta\bar{\rho}}{2\sqrt{N}} S_M\right) \\ G_{-\eta} &= G_\eta^*. \end{aligned} \tag{3.8}$$

The plus sign in the above summation indicates that we only sum over positive values of M_1 and M_2. By using (3.8) to evaluate the two-point function we obtain

$$\langle G_\eta(E)G_{-\eta}(E)\rangle_c = (2\pi\bar\rho)^2 N^{-1/2}\int_0^\infty \mathrm{d}S\,\phi_2(S)\exp[-(\eta_1+\eta_2)S\bar\rho] \tag{3.9}$$

where

$$\phi_2(S) = \sum_{K_1K_2}^{+} A_{K_1}A_{K_2}\delta\left(S-\frac{S_{K_1}}{2\sqrt{N}}\right)\exp[\mathrm{i}\sqrt{N}(S_{K_1}-S_{K_2})]. \tag{3.10}$$

In the slowly varying terms we have put $S_{K_1} = S_{K_2}$. By equating (3.3) and (3.9) and inverting the Laplace transform we obtain Berry's semiclassical sum rule (in a slightly different notation):

$$\phi_2(S)\to N^{1/2}/2\pi \qquad S\to\infty. \tag{3.11}$$

By using (2.11) for the number of levels and using the definition of ϕ_2 we obtain for the two-point function

$$\langle N^2_{sm}(\bar n, N)\rangle_c = \frac{8}{\sqrt{N}}\int_0^\infty \mathrm{d}S\,\phi_2(S)\frac{\sin^2(\frac{1}{2}\bar nS)}{S^2}$$
$$\to \bar n \qquad \bar n\to 0. \tag{3.12}$$

In deriving (3.12) we have put $S_{K_1} = S_{K_2}$ in the slowly oscillating terms. By using the integral transform in (2.9) we recover the result of Berry (1985) for the Δ_3 statistic.

Next we consider the three-point function. As for the two-point function, the main contribution stems from the terms for which the fast-oscillating phases approximately cancel. We will use this to simplify the slowly varying factors. We obtain the following result for the three-point function:

$$\langle G_{\eta_1}(E)G_{-\eta_2}(E)G_{-\eta_3}(E)\rangle_c$$
$$= (-2\pi\mathrm{i}\bar\rho)^3 N^{-3/4}\int_0^\infty \mathrm{d}S\int_0^\infty \mathrm{d}T\,\phi_3(S,T)\exp[-(\eta_1+\eta_3)S\bar\rho-(\eta_2+\eta_3)T\bar\rho] \tag{3.13}$$

where the function $\phi_3(S, T)$ is defined as

$$\phi_3(S,T) = \sum_{K_1K_2K_3}^{+} A_{K_1}A_{K_2}A_{K_3}\delta\left(S-\frac{S_{K_1}}{2\sqrt{N}}\right)\delta\left(T-\frac{S_{K_2}}{2\sqrt{N}}\right)$$
$$\times\exp(\mathrm{i}\sqrt{N}(S_{K_1}-S_{K_2}-S_{K_3})+\tfrac{1}{4}\pi\mathrm{i}). \tag{3.14}$$

By inversion of the Laplace transform we obtain the behaviour of $\phi_3(S, T)$ for large values of S and T:

$$\phi_3(S,T)\to N^{3/4}/4\pi^2 \qquad S,\,T\to\infty. \tag{3.15}$$

This result is a generalisation of Berry's semiclassical rule for the two-point function. It shows that there are intimate correlations between the amplitudes A_M and the phases S_M.

With the help of the sum rule (3.15) we are able to evaluate the short-range behaviour of the three-point function. By using equation (2.11) we obtain

$$\langle N^3_{sm}(\bar{n}, N)\rangle_c = 24N^{3/4}\sum_{K_1K_2K_3}^{+}\frac{A_{K_1}A_{K_2}A_{K_3}}{S_{K_1}S_{K_2}S_{K_3}}\,\mathrm{i}^{-3}$$
$$\times\left\{\exp[\mathrm{i}\sqrt{N}(S_{K_1}-S_{K_2}-S_{K_3}+\tfrac{1}{4}\pi)]\left[\exp\left(\mathrm{i}\frac{\bar{n}}{2\sqrt{N}}S_{K_1}\right)-1\right]\right.$$
$$\left.\times\left[\exp\left(-\mathrm{i}\frac{\bar{n}}{2\sqrt{N}}S_{K_2}\right)-1\right]\left[\exp\left(-\mathrm{i}\frac{\bar{n}}{2\sqrt{N}}S_{K_3}\right)-1\right]+\mathrm{cc}\right\}. \tag{3.16}$$

In the slowly oscillating terms S_{K_1} can be replaced by the sum $S_{K_2}+S_{K_3}$. After substitution of the definition of $\phi_3(S, T)$ (see (3.14)) and a rescaling of the integration variable we obtain the following result for the three-point function:

$$\langle N^3_{sm}(\bar{n}, N)\rangle_c = \frac{12}{\pi^2}N^{-3/4}\int_0^\infty \mathrm{d}S\int_0^\infty \mathrm{d}T\frac{\phi_3(S, T)}{ST(S+T)}\sin[\tfrac{1}{2}\bar{n}(S+T)]\sin(\tfrac{1}{2}\bar{n}S)\sin(\tfrac{1}{2}\bar{n}T). \tag{3.17}$$

For small values of $\bar{n}$ only large values of S and T contribute to the integral.

We can use the asymptotic values of $\phi(S, T)$ in (3.17). After rescaling the variables we obtain

$$\langle N^3_{sm}(\bar{n}, N)\rangle_c = \frac{6}{\pi^2}\bar{n}\int_0^\infty \mathrm{d}S\int_0^\infty \mathrm{d}T\frac{\sin(S+T)\sin(S)\sin(T)}{ST(S+T)}$$
$$= \bar{n} \tag{3.18}$$

where we have used that the integral over S and T is equal to $\frac{1}{6}\pi^2$.

The short-range behaviour of the four-point function can be derived in a similar fashion. In this case we need two sum rules corresponding to the two different locations of the poles of the Green functions in the complex plane. In the case with two poles on either side of the real axis we obtain

$$\langle G_{\eta_1}(E)G_{\eta_2}(E)G_{-\eta_3}(E)G_{-\eta_4}(E)\rangle_c = (2\pi\mathrm{i}\bar{\rho})^4N^{-1}\int_0^\infty \mathrm{d}S\int_0^\infty \mathrm{d}T\int_0^{S+T}\mathrm{d}U$$
$$\times\phi_{41}(S, T, U)\exp\{-\bar{\rho}[(\eta_1+\eta_4)S+(\eta_2+\eta_4)T+(\eta_3-\eta_4)U]\} \tag{3.19}$$

where $\phi_{41}(S, T, U)$ is defined as

$$\phi_{41}(S, T, U) = \sum_{K_1K_2K_3K_4}^{+}A_{K_1}A_{K_2}A_{K_3}A_{K_4}\delta\left(S-\frac{S_{K_1}}{2\sqrt{N}}\right)\delta\left(T-\frac{S_{K_2}}{2\sqrt{N}}\right)\delta\left(U-\frac{S_{K_3}}{2\sqrt{N}}\right)$$
$$\times\exp[\mathrm{i}\sqrt{N}(S_{K_1}+S_{K_2}-S_{K_3}-S_{K_4})]. \tag{3.20}$$

The second case with three poles on one side of the real axis and one on the other leads to the result

$$\langle G_{\eta_1}(E)G_{\eta_2}(E)G_{\eta_3}(E)G_{-\eta_4}(E)\rangle_c = -(2\pi\mathrm{i}\bar{\rho})^4N^{-1}\int_0^\infty \mathrm{d}S\int_0^\infty \mathrm{d}T\int_0^\infty \mathrm{d}U$$
$$\times\phi_{42}(S, T, U)\exp\{-\bar{\rho}[(\eta_1+\eta_4)S+(\eta_2+\eta_4)T+(\eta_3+\eta_4)U]\} \tag{3.21}$$

where ϕ_{42} is defined as

$$\phi_{42}(S, T, U) = \sum_{K_1K_2K_3K_4}^{+}A_{K_1}A_{K_2}A_{K_3}A_{K_4}\delta\left(S-\frac{S_{K_1}}{2\sqrt{N}}\right)\delta\left(T-\frac{S_{K_2}}{2\sqrt{N}}\right)\delta\left(U-\frac{S_{K_3}}{2\sqrt{N}}\right)$$
$$\times\exp[\mathrm{i}\sqrt{N}(S_{K_1}-S_{K_2}-S_{K_3}-S_{K_4})]. \tag{3.22}$$

As in case of the three-point function, we can obtain the asymptotic result for ϕ_{4i} by equating equations (3.19) and (3.21) to equations (3.7*a*) and (3.7*b*), respectively. We just quote the final result:

$$\phi_{41}(S, T, U) \to N/8\pi^3 \qquad S, T, U \to \infty,\ U \in [0, S+T] \tag{3.23a}$$

$$\phi_{42}(S, T, U) \to N/8\pi^3 \qquad S, T, U \to \infty. \tag{3.23b}$$

In equation (3.23*a*) the result for ϕ_{41} is zero when U is outside the range $[0, S+T]$. By using the sum rules we find that for small values of $\bar{n}$ the fourth cumulant converges to $\bar{n}$. Since no new features arise we omit the derivation of this result.

In the derivation of the sum rules we have not used any special properties of the semiclassical expansion of the level density for integrable systems. Consequently very similar sum rules can be obtained for classically chaotic systems.

4. The asymptotic limit of the two-point function

In this section we evaluate the two-, three- and four-point functions from the semiclassical expression for the number of levels (equation (2.9)). For large N and $\bar{n} \sim O(N^\varepsilon)$, ($\varepsilon$ arbitrarily small but larger than zero) we can approximate the product of the fast-oscillating cosines in the two-point function $\langle N^2_m(\bar{n}, N)\rangle_c$ as

$$\left\langle \prod_{i=1}^{2} \cos\left(\sqrt{N} S_{\boldsymbol{K}_i} + \frac{\bar{n}}{4\sqrt{N}} S_{\boldsymbol{K}_i} - \tfrac{1}{4}\pi\right)\right\rangle_c = \tfrac{1}{2}\delta_{\boldsymbol{K}_1 \boldsymbol{K}_2}. \tag{4.1}$$

By using this result we obtain a very simple expression for the two-point function:

$$\Sigma_2(\bar{n}, N) = 64\sqrt{N} \sum_{\boldsymbol{M}}^{+} \frac{A^2_{\boldsymbol{M}}}{S^2_{\boldsymbol{M}}} \sin^2\left(\frac{\bar{n}}{4\sqrt{N}} S_{\boldsymbol{M}}\right). \tag{4.2}$$

(This result has also been obtained by Berry 1986.) For small values of $\bar{n}$ this expression can be evaluated by replacing the summation by an integration from zero to ∞. After rescaling the integration variables we recover the result given in (3.12). However, we have not been careful enough: we should have replaced the summations by an integration from $\frac{1}{2}$ to ∞. The difference can be estimated by putting one of the M_i equal to $\frac{1}{4}$ and replacing the remaining summation by an integration. By rescaling the integration variable we find, for the behaviour of $\Sigma_2(\bar{n}, N)$ for small $\bar{n}$,

$$\Sigma_2(\bar{n}, N) = \bar{n} - \alpha\bar{n}^2/\sqrt{N} \tag{4.3}$$

where α is a numerical constant of order unity. We conclude that the scale at which $\Sigma_2(\bar{n}, N)$ starts deviating from the Poisson limit is given by $\bar{n} \sim N^{1/2}$. By inspection of (4.2) we observe that $\Sigma_2(\bar{n}, N)$ saturates at a scale $\bar{n} \sim N^{1/2}$. Note that the summation over the M_i in (4.2) is not uniformly convergent in $\bar{n}$ so that we are not allowed to differentiate with respect to $\bar{n}$ behind the summation. *A priori* we had not expected that the approximation (4.1) would give the right number variance for small values of $\bar{n}$. For $\bar{n} \sim 1$ the main contribution in the summation comes from a region with $S_{\boldsymbol{M}} \sim \sqrt{N}$. The average distance of neighbouring $S_{\boldsymbol{M}}$ is $1/\sqrt{N}$ and therefore there are non-diagonal contributions on the left-hand side of equation (4.1) even in the limit $N \to \infty$. However, when $\bar{n} \sim N^\varepsilon$, ($\varepsilon > 0$ but arbitrarily small) the approximation in (4.1) is valid in the limit $N \to \infty$ (and also equation (4.3)). Because of (4.3), the linear behaviour of the number variance extends to a much larger region than justified by the derivation for

the semiclassical sum rule in (3.11). Accordingly, the semiclassical sum rule can be extended to a classical sum rule which is valid for much smaller values of S (see Berry and Tabor (1977a, b) and Hannay and Ozorio de Almeida (1984)).

In the study of the Δ_3 statistic by Berry (1985) and Seligman and Verbaarschot (1987) the deviation from the linear behaviour appeared to be much less than that given by (4.3). This is obvious because, under the integral transform that relates the number variance and the Δ_3 statistic, the $\sim\bar{n}^2$ term is transformed to zero (see (2.9) and (2.10)). All higher-order corrections to the Δ_3 statistic will be suppressed by large numerical factors.

For the evaluation of the three-point function $\langle N^3_{sm}(\bar{n}, N)\rangle_c$ we have to deal with the product of three fast-oscillating cosines. For large N only the terms for which the large phases cancel contribute:

$$\left\langle \prod_{i=1}^{3} \cos\left(\sqrt{N}S_{\boldsymbol{K}_i} + \frac{\bar{n}}{4\sqrt{N}} S_{\boldsymbol{K}_i} - \tfrac{1}{4}\pi\right)\right\rangle_c = \delta_{\boldsymbol{K}_1, \boldsymbol{K}_2+\boldsymbol{K}_3}. \tag{4.4}$$

The δ function in (4.4) implies that

$$\boldsymbol{K}_1 = k(\mu_1, \mu_2) \qquad \boldsymbol{K}_2 = l(\mu_1, \mu_2) \qquad \boldsymbol{K}_3 = (k+l)(\mu_1, \mu_2) \tag{4.5}$$

where k, l, m, μ_1 and μ_2 are integers with μ_1 and μ_2 relatively prime. By using (4.4) we can immediately write down the asymptotic result for the three-point function:

$$\Sigma_3(\bar{n}, N) = 192\sqrt{2}N^{3/4} \sum_{\boldsymbol{K}_1\boldsymbol{K}_2\boldsymbol{K}_3}'' \prod_{i=1}^{3} \frac{A_{\boldsymbol{K}_i}}{S_{\boldsymbol{K}_i}} \sin\left(\frac{\bar{n}}{4\sqrt{N}} S_{\boldsymbol{K}_i}\right) \tag{4.6}$$

where the double-primed summation is subject to the constraints in (4.5). From its $\bar{n}$ and N dependence it is clear that the asymptotic formula for $\Sigma_3(\bar{n}, N)$ cannot converge to the Poisson limit for small $\bar{n}$. The only N-independent result that can be obtained from (4.6) is $\Sigma_3(\bar{n}, N) \sim \bar{n}^{3/2}$. This is not in agreement with (3.19) and therefore the diagonal approximation (4.4) is not valid for small $\bar{n}$.

We can estimate at which scale (4.4) is valid. For small $\bar{n}$ the main contributions to (4.6) stem from terms with $S_{\boldsymbol{K}} \sim \sqrt{N}/\bar{n}$. When we use that the number of terms $S_{\boldsymbol{K}_1} + S_{\boldsymbol{K}_2} \in [S, S+\mathrm{d}S]$ is proportional to S^3 we find the condition $\bar{n} \gg N^{1/3}$.

For the four-point function we need the average of four fast-oscillating cosines. The result, which does not include terms for which the arguments of the cosines are not pairwise equal, is

$$\left\langle \prod_{i=1}^{4} \cos\left(\sqrt{N}S_{\boldsymbol{K}_i} + \frac{\bar{n}}{4\sqrt{N}} - \tfrac{1}{4}\pi\right)\right\rangle_c$$
$$= \tfrac{3}{8}\delta_{\boldsymbol{K}_1+\boldsymbol{K}_2, \boldsymbol{K}_3+\boldsymbol{K}_4} + \tfrac{1}{2}\cos(\tfrac{1}{2}\pi)\delta_{\boldsymbol{K}_1, \boldsymbol{K}_2+\boldsymbol{K}_3+\boldsymbol{K}_4} - \tfrac{3}{8}\delta_{\boldsymbol{K}_1, \boldsymbol{K}_2}\delta_{\boldsymbol{K}_2, \boldsymbol{K}_3}\delta_{\boldsymbol{K}_3, \boldsymbol{K}_4}. \tag{4.7}$$

The term with the negative sign is a consequence of the non-zero cumulant of the cosine. The factor $\cos(\frac{1}{2}\pi) = 0$ results from the phase $\frac{1}{4}\pi$ on the left-hand side of this equation. Consequently, the term containing this factor will not contribute to the four-point function. The constraint implicated by the first δ function in (4.7) can be parametrised by the integers k, l, m, μ_1 and μ_2 (the latter two integers are relatively prime) as

$$\begin{aligned} &\boldsymbol{K}_1 = k(\mu_1, \mu_2) \qquad \boldsymbol{K}_2 = l(\mu_1, \mu_2) \qquad \boldsymbol{K}_3 = m(\mu_1, \mu_2) \\ &\boldsymbol{K}_4 = n(\mu_1, \mu_2) \qquad k+l = m+n \qquad k \neq m. \end{aligned} \tag{4.8}$$

The last condition in (4.8) eliminates the correlations that can be written as the product of two two-point functions. The connected part of the four-point function follows

immediately from (4.7). The result is

$$\Sigma_4(\bar{n}) = 1536N\left[\sum_{K_1K_2K_3K_4}^{'''}\prod_{i=1}^{4}\frac{A_{K_i}}{S_{K_i}}\sin\left(\frac{\bar{n}}{4\sqrt{N}}S_M\right) - \sum_{K}^{+}\frac{A_K^4}{S_K^4}\sin^4\left(\frac{\bar{n}}{4\sqrt{N}}S_K\right)\right] \tag{4.9}$$

where the triple-primed summation refers the parametrisation given in (4.8). For the same reasons as in the case of the three-point function, the diagonal approximation (4.8) does not yield the correct short-range behaviour of $\Sigma_4(\bar{n}, N)$. Only for $\bar{n} \gg N^{3/8}$ de we expect this approximation to be valid.

In the next section we will present numerical results for the two-, three- and four-point measures.

5. Numerical results

In our numerical calculations we proceed as follows. From equation (2.2) we generate a level sequence and order it. After unfolding it according to the prescription (2.7) we count the number of levels in consecutive intervals of length $\bar{n}$. By averaging over a part of the spectrum to be specified later on we obtain the lowest four cumulants $\Sigma_p(\bar{n}, N)$. This calculation becomes very time consuming for large values of $\bar{n}$. In order to obtain statistically reliable results we need a huge number of levels. The full circles in figures 1-6 represent the results from these calculations. The statistical errors, which have not been drawn, are equal to the statistical errors of the open circles.

On the other hand we can calculate the cumulants $\Sigma_p(\bar{n}, N)$ from the semiclassical approximation (equation (2.11)) for $\mathcal{N}_N(\bar{n})$. We calculate $N_{sm}(\bar{n}, N)$ in consecutive intervals of length $\bar{n}$ and obtain the moments by averaging over a part of the spectrum. This calculation becomes very time consuming for small values of $\bar{n}$ because we have to use a very large cutoff for the summation over the lattice of integers. The results from these calculations are represented by the open circles in figures 1-7. The statistical errors have been calculated from the fluctuations of moments in neighbouring parts of the spectrum. In this way the systematic error due to the non-stationarity is eliminated.

The short-range behaviour of the cumulants $\Sigma_2(\bar{n}, N)$, $\Sigma_3(\bar{n}, N)$ and $\Sigma_4(\bar{n}, N)$ is shown in figures 1, 2 and 3, respectively. The ratio of α and β is (see equation (2.2)) in all cases equal to $\frac{1}{3}\pi$. We show results for the square well (sw) and the homogeneous x^4 potential. The result given by the full and open circles have been obtained from the 5000th up to the 15 000th level above the ground state. They coincide within the error bars. The lines in figures 1-3 are the results for a random sequence of levels (the Poisson limit). The curves in figures 2 and 3 represent the asymptotic result for the number moments as derived in § 4 with the help of a diagonal approximation (see (4.2), (4.6) and (4.9)). Only for Σ_2 do we obtain the right answer in the semiclassical limit with the diagonal approximation. As argued before, the diagonal approximation does not give the right result for the short-range part of the higher cumulants. We also observe that even at a distance of only one level spacing the fluctuations are less than the Poisson limit. The square well differs more from the Poisson limit than the x^4 potential. This is in agreement with the results of Casati *et al* (1985), Feingold (1985) and Seligman *et al* (1986) for the nearest-neighbour spacing distribution. In the semiclassical calculations the summation over the integers has been cut off at 600. To reduce the computation time we have extrapolated from the results with a cutoff at 400, 500 and 600.

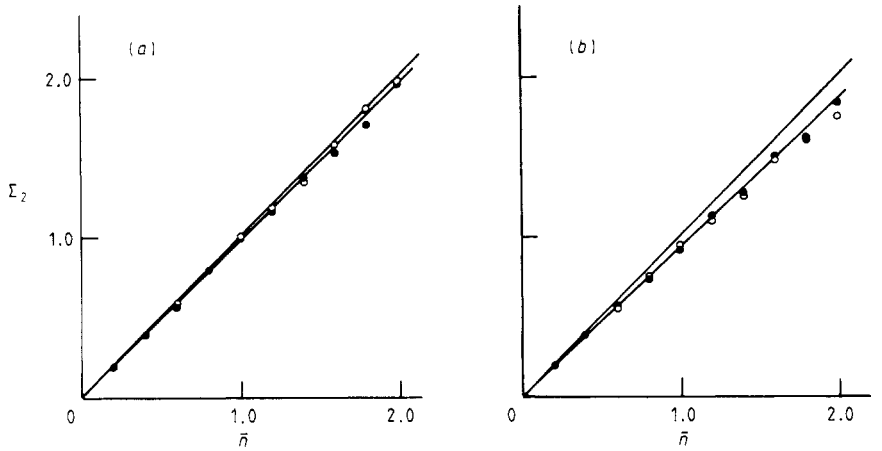

Figure 1. The short-range behaviour of the number variance $\Sigma_2(\bar{n}, N)$ as a function of the average number of levels $\bar{n}$ in a given interval. We show results for (a) the homogeneous x^4 potential and for (b) the square well (SW). The ratio of α and β is equal to $\frac{1}{3}\pi$. The full circles represent the results obtained from the exact level sequence and the open circles the results obtained from the semiclassical expression (2.11) for the 5000th up to the 15 000th level. The Poisson limit is given by the upper line and the asymptotic limit for $\Sigma_2(\bar{n}, N)$ (see equation (4.2)) by the lower one.

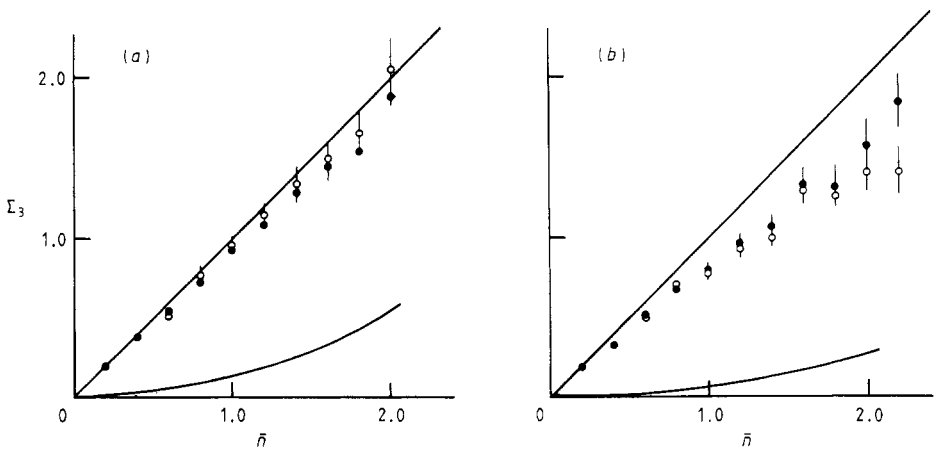

Figure 2. The short-range behaviour of the third cumulant $\Sigma_3(\bar{n}, N)$ as a function of the average number of levels $\bar{n}$ in a given interval. In this figure the curve is obtained from the asymptotic result for the three-point function given by (4.6). For further explanation see the caption of figure 1.

The long-range behaviour of the cumulants $\Sigma_2(\bar{n}, N)$, $\Sigma_3(\bar{n}, N)$ and $\Sigma_4(\bar{n}, N)$ is shown in figures 4, 5 and 6, respectively. The homogeneous x^4 potential and the square well have been investigated for ratios of α and β equal to $\frac{1}{3}\pi$ and $\frac{1}{2}(\sqrt{5}+1)$. The meaning of the full and open circles has already been discussed. They have been obtained from a spectral average over the 500 000th up to the 1500 000th level and coincide within the error bars. In this case the semiclassical calculations have been performed with a cutoff of 120. These results are compared to the asymptotic limit of the number moments $\Sigma_p(\bar{n}, N)$ (see (4.2), (4.6) and (4.9)). The broken curves have

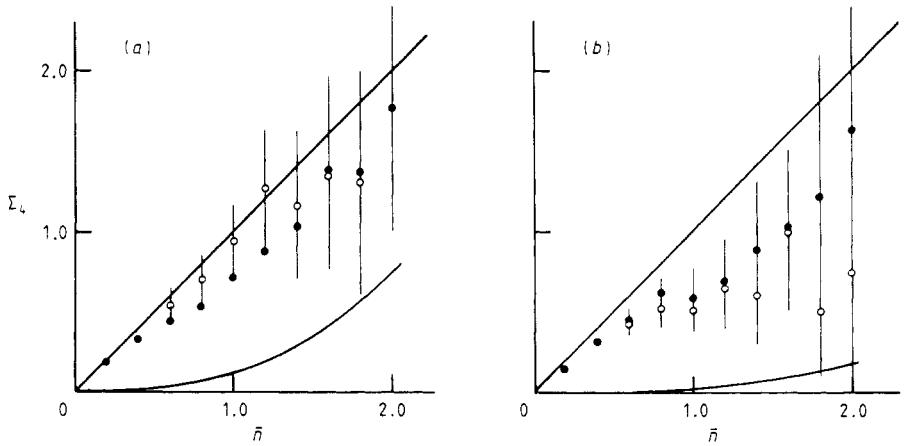

Figure 3. The short-range behaviour of the fourth cumulant of the number of levels $\Sigma_4(\bar{n}, N)$ as a function of the average number of levels $\bar{n}$ in a given interval. In this figure the curve is obtained from the asymptotic result for the four-point function given in (4.9). For further explanation see the caption of figure 1.

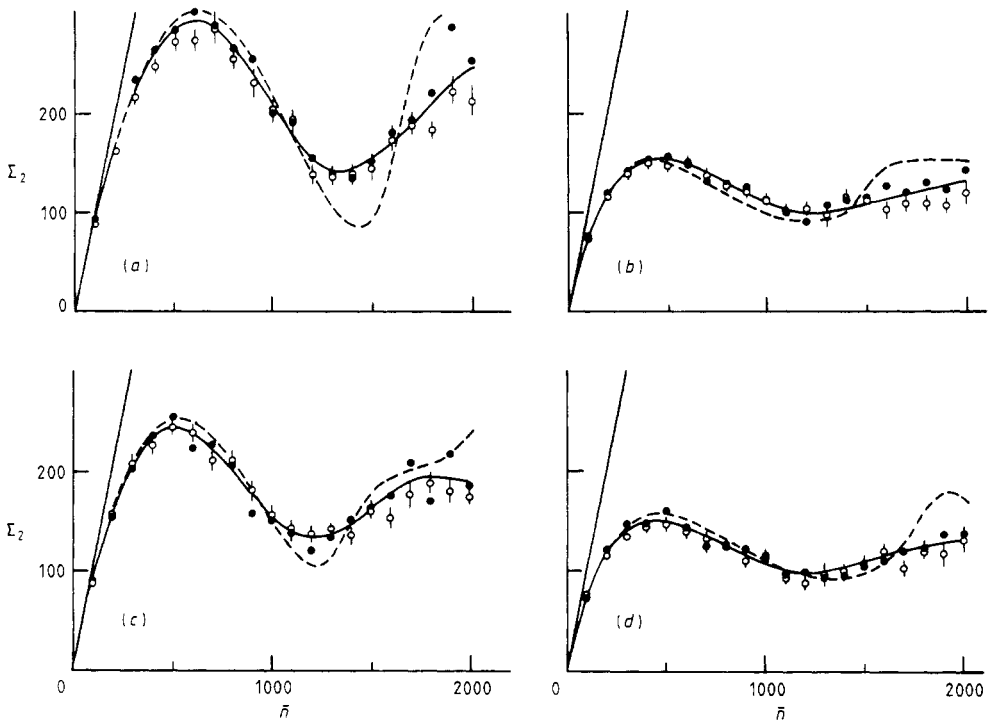

Figure 4. The long-range behaviour of the number variance $\Sigma_2(\bar{n}, N)$ as a function of the average number of levels $\bar{n}$ in a given interval for (*a*), (*c*) the x^4 potential and (*b*), (*d*) the square well (SW). The ratio of α and β is equal to $\frac{1}{3}\pi$ (*a*), (*b*) and $\frac{1}{2}(\sqrt{5}+1)$(*c*), (*d*). The full circles (results from the exact level sequence) and the open circles (results from (2.11)) have been obtained from the 500 000th up to the 1500 000th level above the ground state. The broken curve gives the asymptotic result for the number variance (equation (4.2)) for $N = 10^6$ and the full curve the spectral average of the asymptotic result. The full line represents the Poisson limit.

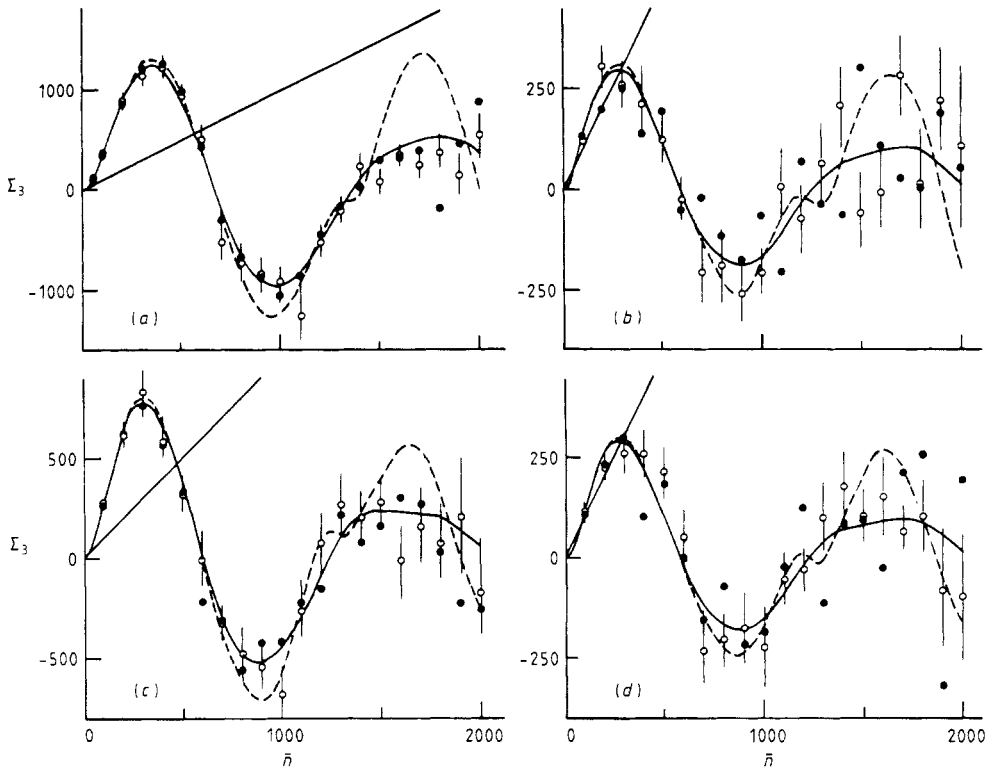

Figure 5. The long-range behaviour of the third cumulant $\Sigma_3(\bar{n}, N)$ as a function of the average number of levels $\bar{n}$ in a given interval. In this figure the broken curves and the full curves have been obtained from the asymptotic result for the three-point function (equation (4.6)). For further explanation see the caption of figure 4.

been calculated for a fixed value of $N = 1000\,000$ and the full curves from an average over ten equidistant values of N in the range [500 000; 1500 000]. The fourth moment requires further discussion. In the exact calculation we subtract three times the square of the ensemble averaged second moment from the fourth moment. In the calculation of the asymptotic result we automatically subtract three times the square of the local average $\Sigma_{2i}(\bar{n}, N)$ of the second moment (by the parametrisation given in (4.8)). Because of this we have to add

$$\frac{3}{k^2}\sum_{i,j=1}^{k}\Sigma_{2i}(\bar{n}, N)\Sigma_{2j}(\bar{n}, N)-\frac{3}{k}\sum_{i=1}^{k}\Sigma_{2i}^2(\bar{n}, N) \tag{5.1}$$

to the semiclassical result (in our case k is equal to 10) in order to compare to the other data. This correction has been included in the full curves in figures 4, 5 and 6. In all cases we find perfect agreement of the asymptotic formula and the other data.

In figure 7 we show the semiclassical result for the four-point function obtained from the 0.5×10^8th up to the 1.5×10^8th level above the ground state of an x^4 potential with the ratio of α and β to $\frac{1}{3}\pi$. The curve represents the asymptotic expression for the four-point function at $N = 10^8$. We obtain a perfect description of the data generated from (2.11) and the asymptotic formula (4.9) without a spectral average and without the correction term given in (5.1). Due to numerical limitations we are not able to present results obtained from the exact level sequence.

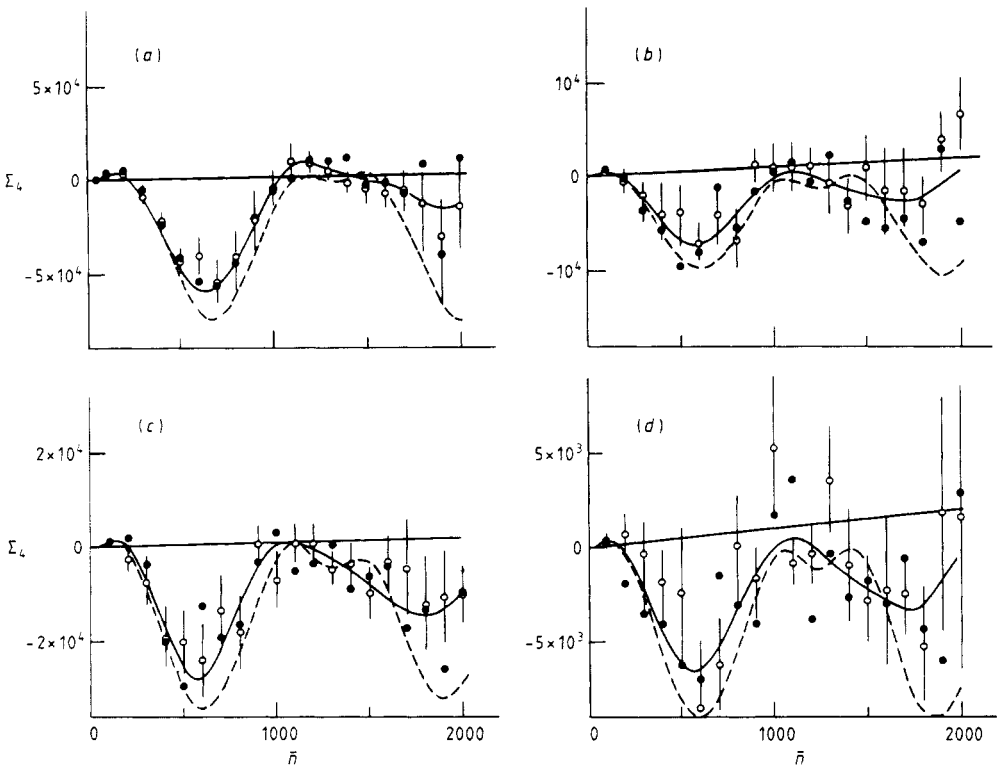

Figure 6. The long-range behaviour of the fourth cumulant $\Sigma_4(\bar{n}, N)$ as a function of the average number of levels $\bar{n}$ in the interval. In this figure the broken curves and the full curves have been obtained from the asymptotic result for the four-point function (equation (4.9)). For further explanation see the caption of figure 4.

6. Conclusions

The basic conclusion of this paper is that the higher-order cumulants of the distribution of the number of levels in a given interval show large deviations from the Poisson limit. The three- and four-point cumulants of the number of levels in intervals of intermediate length show much larger fluctuations than in the Poisson limit. For intervals of asymptotically large length the magnitude of the fluctuations saturates. This has been supported by both analytical and numerical evidence from scale invariant systems. For arbitrary bound systems with a Hamiltonian given by the sum of a kinetic and potential energy we expect a similar behaviour; the structure of the semiclassical expansion does not depend on the details of the integrable system. Moreover, when the potential is a polynomial only the highest-order monomials are of importance in the region far above the ground state.

As in the case of the Δ_3 statistic, the short-range behaviour of cumulants could be obtained with the help of semiclassical sum rules. In deriving these sum rules we have not used any details of the semiclassical periodic orbit sum. Therefore similar sum rules will be valid for the expansion coefficients and the phases of non-integrable systems.

We have also obtained explicit expressions for the long-range behaviour of the second, third and fourth cumulants by taking into account only those terms for which

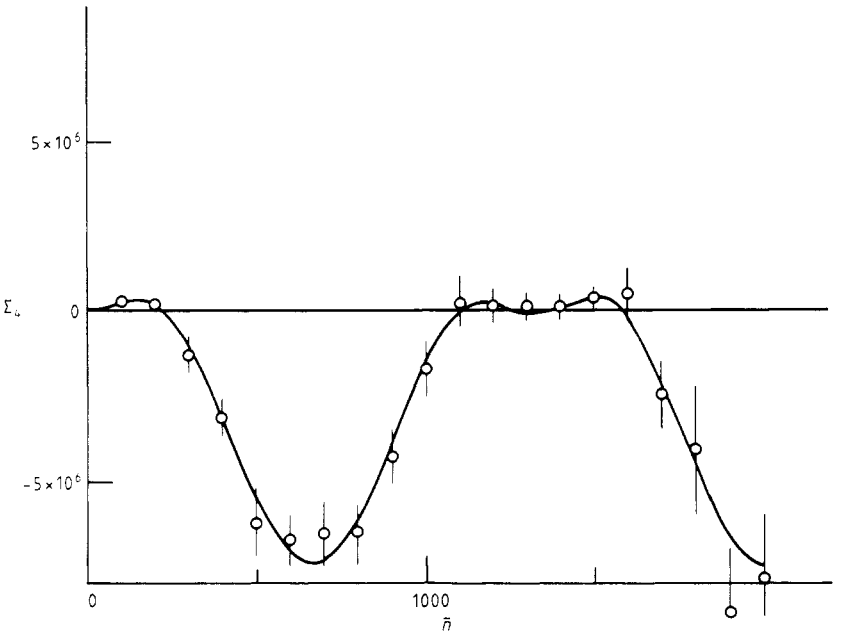

Figure 7. The long-range behaviour of the fourth cumulant $\Sigma_4(\bar{n}, N)$ as a function of the average number of levels $\bar{n}$ in a given interval for the x^4 potential with the ratio of α and β equal to $\frac{1}{3}\pi$. The open circles have been obtained from the semiclassical result (2.11) by using results for 0.95×10^8th level up to the 1.05×10^8th level. In this figure the curve is obtained from the asymptotic result for the four-point function at $N=10^8$ (see (4.9)).

the fast-oscillating exponents cancel. This procedure is not valid when the length of the interval $\bar{n}$ (in units of the average level spacing) is small. The result for the second cumulant is not valid below a scale of $\bar{n}\sim 1$ and the result for the third and fourth cumulants is not valid below a scale of $\bar{n}\sim N^{1/3}$ and $\bar{n}\sim N^{3/8}$, respectively (N is the distance above the ground state measured in units of the average level spacing). From the explicit expressions we have reached the following conclusions. The scale at which the fluctuations no longer increase is given by $\bar{n}\sim\sqrt{N}$. The Poisson limit is well approximated in a region with $\bar{n}\ll N^{1/2}$. The deviation of the second cumulant from the Poisson limit is proportional to $-\bar{n}^2/\sqrt{N}$. For the higher-order cumulants we expect similar corrections. The quadratic correction is not present in the Δ_3 statistic. The reason is that the Δ_3 statistic is related to the number variance by a linear integral transform with a non-trivial kernel. All these results have been confirmed by numerical evidence.

The asymptotic results for higher-order correlation functions might be of importance in the study of molecular spectra where a large number of levels can be obtained (see, e.g., Levandier *et al* 1986). In particular, the large fluctuations in the three- and four-point functions might be helpful to distinguish integrable from chaotic systems.

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